

## Asymptotic Output Tracking Control of a Class of Linear Systems by Finite-and-Quantized Output Feedback\*

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This paper presents a novel finite-and-quantized output feedback asymptotic tracking control method for a general class of continuous-time linear time-invariant systems. First, we construct a finite quantizer with time-varying thresholds and design a pole placement control law that exclusively utilizes the finite-and-quantized output signal and an external reference signal. Then, we establish the boundedness of all closed-loop signals and prove the asymptotic convergence of the output tracking error to zero. The proposed method combines the advantages of classical pole placement control technique and finite quantization feedback technique. It not only reduces the requirement for feedback information compared with existing tracking control methods but also effectively handles unstable poles and zeros in controlled systems, thereby achieving asymptotic output tracking. Finally, we provide a representative example to validate the effectiveness and new features of our proposed method.

**Keywords:** Continuous-time; asymptotic output tracking; pole placement control; quantized-output feedback.

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### 1. Introduction

The field of quantized control systems encompasses control systems that are subject to constraints on quantization or saturation, making them highly relevant in domains such as digital control systems, hybrid systems, and networked control systems [1, 2]. This has led to significant attention

being devoted to their critical applications. The occurrence of measurement errors and saturation constraints in real control systems is attributed to the inherent limitations of sensor accuracy and magnitude, which are inevitable in practical applications. The utilization of quantization and saturation feedback control technique is a potent approach to address the aforementioned issue, and it offers distinct advantages [1, 2]. For example, the study conducted by [3] highlights that quantized and saturated feedback control methods exhibit greater robustness against external disturbances and/or measurement errors compared with exact feedback control. A fundamental problem in quantized control is how to jointly design feedback controllers

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and quantizers to achieve a given control objective [1]. The application of classical state and output feedback control methods to such control systems is often limited, particularly when dealing with finite quantization. Therefore, the systematic investigation of quantized control systems holds substantial theoretical and practical significance.

The quantizers utilized in control systems are instrumental devices that discretize continuous signals, exerting a profound influence on the performance and stability of the control systems. The categorization of the quantizers is typically based on whether the quantizer is static or dynamic. Static quantizers are usually easy to employ, but may be difficult to achieve global convergence and often require infinite quantization levels [4–7]. Aiming to address the unresolved problems encountered in static quantizers, Brockett and Liberzon for the first time proposed a class of dynamic quantizers in [8]. The dynamic quantizers, in contrast to static quantizers, typically incorporate an adjustable parameter known as “sensitivity” which holds promising practical applications such as vision-based control. A comprehensive description of these applications can be found in [8]. So far, the two types of quantizers have been extensively utilized in diverse control systems, and we have witnessed remarkable advancements in the theory and applications of quantized feedback control [2, 6, 7, 9–17, 23, 19–25]. For example as early as 1956, authors in [26] studied quantization effects in sampled data systems. In the 1960s, [27] proposed the idea of quantization control to reduce the computational burden of optimal design. Since then, various quantization control methods and applications have evolved. For instance, [28] studied quantized control systems in the framework of discontinuous control systems. Reference [29] proposed a Lyapunov–Krasovskii generalization method to solve the quantized feedback problem with delay. The sliding mode control problem with quantized or saturated signals is researched in [30–33]. The problem of achieving coherence or formation of multi-agent systems through quantized communication is studied in [34–36]. The problem of feedback control for networked systems with distributed time delays is discussed in [37]. The control problem for systems affected by saturation is involved in [38–41].

Among these quantized feedback control methods, some works aim to address the finite-and-quantized output feedback tracking control problems under a model reference control framework. For instance, [42] studied the finite-and-quantized output tracking for discrete-time Linear Time-Invariant (LTI) minimum-phase systems. In general, the model reference control method is commonly used to solve the tracking control problem. However, this method is only applicable to minimum-phase systems [43]. Therefore, in order to effectively control general class of LTI systems covering both minimum-phase and nonminimum-phase

systems, [44, 45] combine the dynamic quantization technique and the Pole Placement Control (PPC) technique to solve the output tracking control problem of LTI systems using finite-and-quantized output feedback. In these works, the proposed control laws (see, for example, [42, 44, 45]) generally realize the output tracking errors converge to some residual sets of the origin. The technical problem is formulated by the following example. Consider an LTI system model:  $P(s)y(t) = Z(s)u(t), t \geq t_0$ , where  $s$  is the differentiation operator,  $t_0$  is the initial moment,  $P(s) = s^n + p_{n-1}s^{n-1} + \dots + p_1s + p_0$ , and  $Z(s) = z_{n-1}s^{n-1} + \dots + z_1s + z_0$ . Let  $y^*(t)$  denote a bounded and time-varying reference signal [44]. Define the floor function  $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k < x\}$ . The dynamic quantizer is chosen as

$$q(y(t), \Delta(t)) = \begin{cases} M & \text{if } y(t) > \left(M + \frac{1}{2}\right) \Delta(t), \\ \left\lfloor \frac{y(t)}{\Delta(t)} + \frac{1}{2} \right\rfloor \Delta(t) & \text{if } -\left(M + \frac{1}{2}\right) \Delta(t) < y(t) \leq \left(M + \frac{1}{2}\right) \Delta(t), \\ -M & \text{if } y(t) \leq -\left(M + \frac{1}{2}\right) \Delta(t), \end{cases} \quad (1)$$

where  $M \in \mathbb{Z}$  is a positive integer and  $\Delta(t)$  is a time-varying sensitivity to be designed. By employing a quantized-output feedback signal  $\Delta(t)q(y(t), \Delta(t))$  and the reference signal  $y^*(t)$ , [44] devises a finite-and-quantized output feedback control law that yields a closed-loop tracking error equation satisfying  $e(t) = \frac{Z(s)D(s)}{A^*(s)}[e_q(y, \Delta)](t) + \epsilon(t)$ , where  $e_q(y(t), \Delta(t)) = \Delta(t)q(y(t), \Delta(t)) - y(t)$ ,  $D(s)$  and  $A^*(s)$  are some polynomials of appropriate degrees and  $\epsilon(t)$  is some exponentially decaying signal. The convergence of the tracking error  $e(t)$  to zero necessitates the simultaneous convergence of the quantization error  $e_q(y(t), \Delta(t))$  to zero. As shown in [44],  $e_q(y(t), \Delta(t))$  satisfies  $|e_q(y(t), \Delta(t))| \leq \frac{1}{2} \Delta(t)$ . To achieve asymptotic tracking,  $\Delta(t)$  should converge to zero. Note that the quantization amplitude  $M$  is finite, which indicates  $y(t)$  converges to zero if  $\Delta(t)$  converges to zero and the quantizer is unsaturated. The presence of a nonzero reference signal  $y^*(t)$  leads to a contradiction, thus concluding that the aforementioned method cannot achieve asymptotic tracking.

After conducting a comprehensive literature review, it becomes evident that the investigation of achieving asymptotic output tracking solely through finite-and-quantized output feedback for a general class of LTI systems (including continuous-time and discrete-time) remains an open area for further study. This paper will systematically address this problem. Particularly, [46] studied the problem of parameter estimation for finite impulse response systems with binary observations, where a new time-varying threshold design is

proposed and the parameter estimation algorithm is convergent. This result motivates us to consider whether the quantizers with time-varying thresholds can be employed to solve the problem addressed in this paper. Overall, combining the dynamic quantization technique in [8], the PPC technique in [43] and the quantizer design with time-varying thresholds in [46], this paper will give a thorough study on the asymptotic output tracking problem for a general class of continuous-time LTI systems by using finite-and-quantized output feedback. The contributions of this paper are as follows:

- A finite-and-quantized output feedback PPC law is proposed for a general class of continuous-time LTI systems, encompassing both minimum phase and nonminimum phase systems. The proposed PPC law incorporates a novel dynamic quantizer with time-varying thresholds which plays a crucial role in ensuring the stability and asymptotic output tracking performance of the closed-loop system.
- Compared with the existing literature, the proposed control method has its unique advantages: (i) the proposed control law is constructed by means of only an external reference output and a designed finite-and-quantized output feedback; (ii) the controlled system is allowed to have unstable poles and zeros, i.e. the proposed method is valid for both minimum-phase and nonminimum-phase systems; and (iii) the proposed control law guarantees asymptotic output tracking under the limitation of only having access to finite-and-quantized output feedback.

The remainder of this paper is organized as follows. The last of Sec. 1 introduces the notation employed, and Sec. 2 provides the problem statement and the preliminaries including system model, quantizer model, reference output model, control objective and assumption. Section 3 is the main part of this paper presenting the quantized-output feedback control design and the corresponding theoretical results. Section 4 presents a typical simulation example to illustrate the performance of the proposed method. Finally, Sec. 5 concludes this paper and gives issues that could be considered in the future.

**Notation.** In this paper, we use  $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}$  and  $\mathbb{Z}^+$  to denote the sets of real numbers, positive real numbers, integers and positive integers, respectively. Let  $s$  denote the differentiation operator, i.e.  $sx(t) = \dot{x}(t)$  with  $x(t) \in \mathbb{R}^n$ ,  $t \geq t_0$ . By  $L^\infty$  and  $L^1$ , we denote two signal spaces defined as  $L^\infty = \{x(t) : \|x(\cdot)\|_\infty < \infty\}$  and  $L^1 = \{x(t) : \|x(\cdot)\|_1 < \infty\}$  with  $\|x(\cdot)\|_\infty = \sup_{t \geq t_0} \|x(t)\|_\infty$  and  $\|x(\cdot)\|_1 = \int_{t_0}^\infty \|x(t)\|_1 dt$ , respectively. In particular, we use the notation:  $y(t) = G(s)[u](t)$  to denote the output  $y(t)$  of a continuous-time LTI system represented by a transfer function  $G(s)$  with input  $u(t)$ . This notation not only fuses time-domain and  $s$ -domain signal operations, but

also provides a demonstration of the control system while eliminating the need for complex convolutional expressions, and they are beneficial for control design and analysis.

## 2. Problem Statement

In this section, we introduce the system model to be analyzed and present the problem to be addressed.

**System model.** Consider the following continuous-time Single-Input and Single-Output (SISO) LTI system:

$$P(s)y(t) = Z(s)u(t), \quad t \geq t_0, \quad (2)$$

where  $y(t) \in \mathbb{R}$ ,  $u(t) \in \mathbb{R}$  are the output and the input, respectively,  $t_0$  is the initial moment of the system, and  $P(s)$  and  $Z(s)$  are some polynomials with constant coefficients and of degrees  $n$  and  $n - 1$ , respectively, i.e.

$$\begin{aligned} P(s) &= s^n + p_{n-1}s^{n-1} + \cdots + p_1s + p_0, \\ Z(s) &= z_{n-1}s^{n-1} + z_{n-2}s^{n-2} + \cdots + z_1s + z_0. \end{aligned}$$

Since  $y(t)$  in the system model (2) cannot be measured precisely in some real-world scenarios, we consider the case where only the quantized value of  $y(t)$  can be obtained for feedback control. However, unlike the quantizer (1) in [8], we design a dynamic quantizer with time-varying thresholds. In this paper, we use  $Q(y(t), \Delta(t), y^*(t))$  to denote the quantizer, where  $\Delta(t)$  is a time-varying signal called the sensitivity of the quantizer.

**Quantizer model.** In this paper, a new class of time-varying threshold quantizers is devised by incorporating the concept of time-varying thresholds from [46] into the quantizers introduced in [8]. It contains three parameters: a constant parameter  $M \in \mathbb{Z}^+$ , a time-varying parameter  $\Delta(t) \in \mathbb{R}^+$ , and the reference signal  $y^*(t)$ . The first two parameters are referred to as the saturation value and the quantization sensitivity, respectively. The output  $y(t)$  of the system is measured by a sensor equipped with  $2M$  thresholds, denoted as  $C_i$ , where  $i = 1, 2, \dots, 2M$ , and satisfying the order relationship:  $-\infty < C_1 < C_2 < \cdots < C_{2M} < \infty$ . The measurements of this sensor can be expressed by  $2M + 1$  indicative functions as follows:

$$Q(y(t), \Delta(t), y^*(t)) = \sum_{i=0}^{2M} (i - M) I_{\{C_i < y_k \leq C_{i+1}\}} = \begin{cases} -M & \text{if } y_k \leq C_1, \\ -(M - 1) & \text{if } C_1 < y_k \leq C_2, \\ \vdots & \vdots \\ M & \text{if } y_k > C_{2M}, \end{cases} \quad (3)$$

where  $C_0 = -\infty$ ,  $C_{2M+1} = \infty$  and for  $i = 1, \dots, 2M$ ,

$$C_i = \Delta(t)(i - M - 1/2) + y^*(t). \tag{4}$$

This quantizer is associated with the reference signal  $y^*(t)$  and sensitivity  $\Delta(t)$ . For convenience, we will abbreviate the quantized value as  $Q(y(t))$ , without causing any confusion.

**Remark 2.1.** The quantizer employed here utilizes varying sensitivity and thresholds, where the sensitivity and thresholds (excluding the saturation value) are adjusted based on quantization measurements and known signals. Such kind of quantizers can be viewed as a device that consists of a multiplier, an adjustable factor, and an analog-to-digital converter [8]. For example, for a temperature sensor with the limited capabilities described above, one can reasonably assume that the threshold settings are allowed to be adjusted based on known signals. In another example, a camera with limited pixel scaling capability can be modeled as a quantizer with varying sensitivity, varying threshold, and fixed saturation values.

**Reference output model.** The reference model is

$$Q_m(s)[y^*](t) = 0, \tag{5}$$

where  $Q_m(s)$  is a monic polynomial of degree  $n_q$ . For the requirement of the boundedness of  $y^*(t)$ ,  $Q_m(s)$  should satisfy that its zeros are either in the open left-half on the complex plane or on the  $j\omega$ -axis but not repeated.

**Remark 2.2.** In this paper, the reference signal is required to satisfy the condition (5). It is known to us that in the framework of model reference control, the reference signal only needs to be bounded. However, the proposed method is not only capable of solving the control problem for minimum-phase systems, but is also valid for nonminimum-phase systems. It is worth pointing out that the condition (5) for  $y^*(t)$  is not restrictive. For many bounded signals, suitable  $Q_m(s)$  can be chosen to make (5) hold. For example, for  $y^*(t) = a \sin(\sigma t) + b \cos(\sigma t)$  with  $\sigma \neq 0$  and  $a^2 + b^2 \neq 0$ , we can choose  $Q_m(s) = s^2 + \sigma^2$ ; for  $y^*(t) = a_1 \sin(\sigma_1 t) + a_2 \sin(\sigma_2 t) + b_1 \cos(\sigma_1 t) + b_2 \cos(\sigma_2 t)$  with  $\sigma_1 \neq 0, \sigma_2 \neq 0, \sigma_1 \neq \sigma_2$  and  $a_i^2 + b_i^2 \neq 0, i = 1, 2$ , we can choose  $Q_m(s) = (s^2 + \sigma_1^2)(s^2 + \sigma_2^2)$ ; and for  $y^*(t) = ce^{-at}$ ,  $c \neq 0$ , we can choose  $Q_m(s) = s + a$ .

**Control objective.** For any given reference signal  $y^*(t) \in L^\infty$  satisfying (5), the control objective is to find a control law  $u(t)$  that contains a finite-and-quantized output  $Q(y(t))$  and a reference output signal  $y^*(t)$ , and use only these two signals for feedback control design of the system (2) so as to ensure that all the closed-loop signals are bounded and that  $y(t) - y^*(t)$  converges to zero asymptotically.

**Assumption.** The only assumption required for the control design is that

$$(A1): P(s)Q_m(s) \text{ and } Z(s) \text{ are coprime.}$$

For Assumption (A1), it is a standard condition for the classical PPC method for continuous-time LTI systems [43]. It will be proven that the quantized output feedback version of the classical PPC law still work under Assumption (A1). By the way, our method proposed in this paper does not require  $Z(s)$  to be stable. In other words, both  $P(s)$  and  $Z(s)$  are allowed to be simultaneously unstable.

### 3. Quantized Feedback Control Design

This section presents the details on designing a finite-and-quantized output feedback control  $u(t)$  to meet the control objective. The whole procedure consists of three main steps. First, we give a key design equation for computing some parameters of the control law design. Second, we design the finite-and-quantized output feedback control law structure and derive the corresponding tracking error equation. Finally, after specifying all of the parameters and signals in the proposed control law, we analyze the closed-loop stability and output tracking performance.

**Key design equation.** To proceed, we review a key design equation in the classical PPC method. This equation is essential for quantized control law design. Choose a monic stable polynomial  $A^*(s)$  of order  $2n + n_q - 1$ , where  $n_q$  is the degree of the polynomial  $Q_m(s)$  in (5). Under assumption (A1), we can solve the following Diophantine equation

$$C(s)P(s)Q_m(s) + D(s)Z(s) = A^*(s), \tag{6}$$

with respect to  $C(s)$  and  $D(s)$  to find a solution of the form

$$C(s) = s^{n-1} + c_{n-2}s^{n-2} + \dots + c_1s + c_0, \\ D(s) = d_{n_q+n-1}s^{n_q+n-1} + \dots + d_1s + d_0.$$

Under Assumption (A1), the solution of the design equation (6) is unique for any  $A^*(s)$  of degree  $2n + n_q - 1$ , of which the proof can be seen in [43].

**Finite-and-quantized feedback PPC law structure.** Motivated by the standard output feedback PPC law in [43], we design the quantized-output feedback PPC law as

$$u(t) = (\Lambda_c(s) - C(s)Q_m(s)) \frac{1}{\Lambda_c(s)} [u](t) \\ - D(s) \frac{1}{\Lambda_c(s)} [Q(y, \Delta, y^*) \Delta](t), \tag{7}$$

where  $C(s)$  and  $D(s)$  can be calculated by solving the equation (6), and  $\Lambda_c(s)$  is a chosen monic stable polynomial of degree  $n + n_q - 1$ . Since  $A^*(s)$  in (6) is monic and of degree  $2n + n_q - 1$  and  $P(s)$  is monic and of degree  $n$ , we have  $C(s)Q_m(s)$  is always monic and of degree  $n + n_q - 1$  so that  $(\Lambda_c(s) - C(s)Q_m(s)) \frac{1}{\Lambda_c(s)}$  is strictly proper. Thus, there does not exist any algebraic loop in the control law (7).

**Remark 3.1.** With (3) and (4), we see that the signal  $Q(y(t), \Delta(t), y^*(t)) \Delta(t)$  is actually a quantization of the tracking error  $y(t) - y^*(t)$ . This idea is similar to the



paper [47] in which the prediction error rather than the state itself is quantized at the time of information transfer. Such a quantizer design plays an important role in obtaining finite-and-quantized output feedback asymptotic tracking results.

To help readers better understand the control law  $u(t)$ , we express (7) as the following form:

$$u(t) = \theta_1^{*T} \omega_1(t) + \theta_2^{*T} \omega_2(t) + \theta_3^* \omega_3(t). \quad (8)$$

Now, we show how to implement the control law (8) in practice. In other words, we specify the parameters  $\theta_i^*$ ,  $i = 1, 2, 3$ , and clarify how to obtain the signals  $\omega_i(t)$ ,  $i = 1, 2, 3$ . First, we denote

$$\begin{aligned} Q_m(s) &= s^{n_q} + q_{n_q-1}s^{n_q-1} + \dots + q_1s + q_0, \\ \Lambda_c(s) &= s^{n_q+n-1} + \lambda_{n_q+n-2}^c s^{n_q+n-2} + \dots + \lambda_1^c s + \lambda_0^c, \\ a_1(s) &= (1, s, \dots, s^{n_q+n-2})^T, \end{aligned}$$

then the parameters can be calculated as

$$\theta_1^* = \begin{pmatrix} \lambda_0^c \\ \lambda_1^c \\ \vdots \\ \lambda_{n_q+n-2}^c \end{pmatrix} - \begin{pmatrix} c_0 & 0 & \dots & 0 \\ c_1 & c_0 & \dots & 0 \\ c_2 & c_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-1} & \dots & c_0 \\ 1 & c_{n-2} & \dots & c_1 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & c_{n-1} \\ 0 & 0 & \dots & c_{n-2} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n_q-1} \\ 1 \end{pmatrix}, \quad (9)$$

$$\theta_2^* = - \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n_q+n-2} \end{pmatrix} + d_{n_q+n-1} \begin{pmatrix} \lambda_0^c \\ \lambda_1^c \\ \vdots \\ \lambda_{n_q+n-2}^c \end{pmatrix}, \quad (10)$$

$$\theta_3^* = -d_{n_q+n-1}. \quad (11)$$

With the above parameters, we re-write (7) as

$$\begin{aligned} u(t) &= (\Lambda_c(s) - C(s)Q_m(s)) \frac{1}{\Lambda_c(s)} [u](t) \\ &\quad - \frac{D(s) - d_{n_q+n-2}\Lambda_c(s)}{\Lambda_c(s)} [Q(y, \Delta, y^*)\Delta](t) \\ &\quad - d_{n_q+n-1}[Q(y, \Delta, y^*)\Delta](t) \\ &= \theta_1^{*T} \frac{a_1(s)}{\Lambda_c(s)} u(t) + \theta_3^*[Q(y, \Delta, y^*)\Delta](t) \\ &\quad + \theta_2^{*T} \frac{a_1(s)}{\Lambda_c(s)} [Q(y, \Delta, y^*)\Delta](t). \end{aligned} \quad (12)$$

To proceed, we denote

$$\omega_1(t) = \frac{a_1(s)}{\Lambda_c(s)} [u](t), \quad (13)$$

$$\omega_2(t) = \frac{a_1(s)}{\Lambda_c(s)} [Q(y, \Delta, y^*)\Delta](t), \quad (14)$$

$$\omega_3(t) = Q(y(t), \Delta(t), y^*(t))\Delta(t). \quad (15)$$

The signals  $\omega_1(t)$  and  $\omega_2(t)$  can be easily obtained from the following auxiliary systems:

$$\dot{\omega}_1(t) = A_\lambda \omega_1(t) + b_\lambda u(t), \quad (16)$$

$$\dot{\omega}_2(t) = A_\lambda \omega_2(t) + b_\lambda Q(y(t), \Delta(t), y^*(t))\Delta(t), \quad (17)$$

where  $\omega_1(t) \in \mathbb{R}^{n_q+n-1}$ ,  $\omega_2(t) \in \mathbb{R}^{n_q+n-1}$ ,

$$\begin{aligned} A_\lambda &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\lambda_0^c & -\lambda_1^c & -\lambda_2^c & \dots & -\lambda_{n_q+n-3}^c & -\lambda_{n_q+n-2}^c \end{pmatrix}, \\ b_\lambda &= (0, \dots, 0, 1)^T \in \mathbb{R}^{n_q+n-1}. \end{aligned}$$

So far, all of the parameters and signals in (8) are specified. In particular, we also clarify the details about how to obtain the signals  $\omega_1(t)$  and  $\omega_2(t)$ . The control law structure is shown in Fig. 1.

**Tracking error equation.** With Remark 3.1, we define the tracking error and the quantized error as

$$e(t) = y(t) - y^*(t), \quad (18)$$

$$e_Q(y(t), \Delta(t), y^*(t)) = y(t) - y^*(t) - \Delta(t)Q(y(t)), \quad (19)$$

respectively. Now, we give the following lemma which specifies a tracking error equation crucial for the sensitivity design and stability analysis.

**Lemma 3.2.** Under Assumption (A1), the finite-and-quantized output feedback PPC law (8), applied to the system (2), ensures

$$e(t) = \frac{Z(s)D(s)}{A^*(s)} [e_Q(y, \Delta, y^*)](t) + \epsilon(t), \quad (20)$$

for some exponentially decaying signal  $\epsilon(t)$ .

**Proof.** See Appendix A. □

See Appendix A for the proof of Lemma 3.2. Equation (20) implies that if exact output feedback is used, i.e.  $e_Q(y(t), \Delta(t), y^*(t)) = 0$ , then the tracking error  $y(t) - y^*(t)$  will converge to zero exponentially.

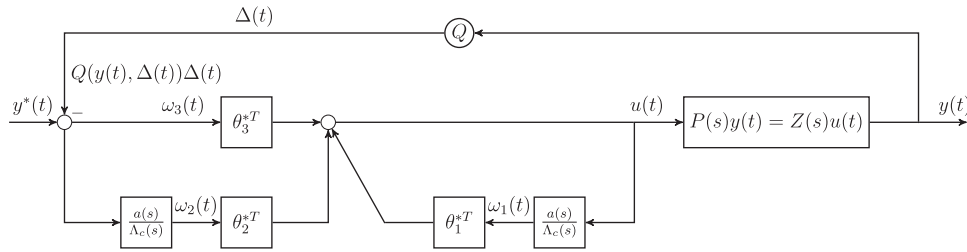


Fig. 1. Finite-and-quantized output feedback control system structure.

**Remark 3.3.** The measurement error in this paper is caused by the quantization of the sensor. From Eq. (19) it is known that this quantization error in turn affects the tracking error. The tracking error in the case of finite-and-quantized output feedback, however, does not decay exponentially. In practice, if the sensitivity is chosen arbitrarily in the controller (12), not only may the tracking target not be realized, but even the system output can diverge to infinity due to feedback saturation.

Next, we show how  $\Delta(t)$  can be designed to meet the control objective.

**Specification of the control law.** The control strategy is usually divided into two stages. First, since the initial output is unknown, we must “zoom out,” i.e. increase  $\Delta(t)$  or choose  $\Delta(t)$  large enough, until the system output can be adequately measured. Second, we will “zooming in” i.e. decrease the sensitivity  $\Delta(t)$  in some way so that it gradually becomes 0.

To specify the control law, we let

$$A^*(s) = \sum_{i=0}^{2n+n_q-1} a_i s^i, Z(s)D(s) = \sum_{i=0}^{n+n_q} b_i s^i,$$

with  $a_{2n+n_q-1} = 1$  and denote

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{2n+n_q-3} & -a_{2n+n_q-2} \end{pmatrix},$$

$$b = (0, \dots, 0, 1)^T \in \mathbb{R}^{2n+n_q-1},$$

$$c^T = (b_0, b_1, \dots, b_{n+n_q}, 0, \dots, 0)^T \in \mathbb{R}^{2n+n_q-1}.$$

(21)

Then (20) can be represented as the following auxiliary system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + be_Q(y(t), \Delta(t), y^*(t)), \\ e(t) &= cx(t). \end{aligned}$$

(22)

The closed-loop system of (22) can be written as

$$\dot{x}(t) = Ax(t) + bQ_s(cx(t)),$$

where  $Q_s(cx(t)) = cx(t) - \Delta(t)Q(y(t))$ .

Noting that  $A$  is stable (due to  $A^*(s)$  is stable), according to the standard Lyapunov stability theory, there exists a positive definite symmetric matrix  $P$  such that

$$A^T P + PA + I = 0. \tag{23}$$

We will let  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  denote the smallest and the largest eigenvalue of a symmetric matrix  $P$ , respectively. Denote  $l = 2n + n_q - 1$ . For an arbitrary  $\delta > 0$ , choose  $M$  large enough to satisfy

$$M > \|c\|(\|Pb\|\sqrt{l} + \delta) \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + 1/2. \tag{24}$$

Then, define the scaling factor by the formula

$$\Omega = \|c\|(\|Pb\|\sqrt{l} + \delta) \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(M - \frac{1}{2}\right)^{-1}. \tag{25}$$

By (24) and the definition of  $\Omega$ , we have  $\Omega < 1$ . Next, define the switching interval  $\tau$  as

$$\tau = \frac{\frac{1}{\|c\|^2} (M - \frac{1}{2})^2 \lambda_{\min}(P) - (\|Pb\|\sqrt{l} + \delta)^2 \lambda_{\max}(P)}{(\|Pb\|\sqrt{l} + \delta)\delta}. \tag{26}$$

By  $\Omega < 1$ , it is easy to conclude  $\tau > 0$ .

**Zooming out stage.** We now describe the “zooming-out” stage of the control strategy. Noting (22), since  $e(t) \in \mathbb{R}$  while  $x(t) \in \mathbb{R}^l$ , the initial “zooming-out” stage in Theorem 1 in [8] cannot be implemented. For this reason, we need *priori* information about the initial value of the system (2) and more specifically, an upper bound on the state  $x(t)$  in the auxiliary system (22).

For the auxiliary system (22), suppose that  $\|x(t_0)\| \leq U$ . The zooming out operation can be done directly at the moment  $t_0$ . Let us choose the initial sensitivity  $\Delta(t_0)$  large enough to have

$$\Delta(t_0) \geq \|c\|U \left(M - \frac{1}{2}\right)^{-1} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}.$$

**Zooming in stage.** With  $\Delta(t_0)$  being specified, we come to the “zooming-in” stage. At this stage, we use the scaling factor

in (25) to reduce the sensitivity  $\Delta(t)$  step by step and use this sensitivity to get a controller applied to the system. The details are as follows. The sensitivity  $\Delta(t)$  is designed to be

$$\begin{aligned} \Delta(t) &= \Omega^k \Delta(t_0), \\ t &\in [t_0 + k\tau, t_0 + (k+1)\tau], \quad \text{for } k = 0, 1, 2, \dots \end{aligned} \quad (27)$$

With this sensitivity design, the control law designed in (8) can be implemented to the system (2).

**Remark 3.4.** To help the readers better understand the proposed PPC strategy, we would like to give the following explanation. Since  $y(t_0)$  cannot be measured, it is not possible to determine whether  $Q(y(t_0))$  is saturated or not. However, since we assume a priori information about the initial value upper bound, we can choose a sufficiently large  $\Delta(t_0)$  at the moment  $t_0$  to make the quantizer unsaturated. Subsequently, we gradually decrease the sensitivity  $\Delta(t)$ , and at the same time utilize the characteristic of the LTI systems to ensure that the quantizer can still be unsaturated after the sensitivity is decreased. This is repeated to realize asymptotic tracking control by finite-and-quantized output feedback. Specific details can be found in the proof of Theorem 3.6.

**Remark 3.5.** The design of the controller (12) is divided into two parts, the controller parameters  $\theta_i^*$ ,  $i = 1, 2, 3$  and the auxiliary signals  $\omega_i(t)$ ,  $i = 1, 2, 3$ , where the controller parameters are obtained directly via (9)–(11). The auxiliary signal is related to the sensitivity  $\Delta$ , where the sensitivity is designed to be related only to the system parameters and the sensor amplitude  $M$ , and can be calculated directly, which are all easy to manipulate in practice.

**System performance analysis.** With the finite-and-quantized output feedback PPC law (8) and sensitivity design (27), we derive the following main result.

**Theorem 3.6.** *Under Assumption (A1), if there exists an upper bound  $U$  such that  $\|x(t_0)\| < U$  with  $x(t)$  defined in (22), the finite-and-quantized output feedback PPC law (7) with the sensitivity  $\Delta(t)$  designed in (27), applied to the system (2) with unmeasurable  $y(t_0) \in \mathbb{R}$ , ensures that all closed-loop signals are bounded and  $\lim_{t \rightarrow \infty} (y(t) - y^*(t)) = 0$ .*

**Proof.** See Appendix B.  $\square$

So far, we have proposed a finite-and-quantized output feedback PPC method. Theorem 3.6 gives an analytical solution to the finite-and-quantized output feedback output tracking control problem for a general class of continuous-time LTI systems which may have unstable zeros and poles.

#### 4. Simulation Study

This section demonstrates the design procedure and validates the effectiveness of the proposed control method.

In the simulation, we use the Runge–Kutta iterative format. The sensitivity values are taken to shrink every  $\tau$  time interval (the values of  $\tau$  are given later). The parameters and auxiliary signals in the controller can be computed directly from the system model.

**Simulation system model.** For the system (2), we choose

$$P(s) = \left(s - \frac{1}{2}\right) \left(s + \frac{3}{2}\right), \quad Z(s) = s - 1.$$

The model exhibits instability due to an unstable pole at  $s = \frac{1}{2}$  and nonminimum phase behavior caused by an unstable zero at  $s = 1$ . For the reference output, we choose  $y^*(t) = \sin(t) + \cos(t)$ , and the corresponding  $Q_m(s)$  can be taken as  $Q_m(s) = s^2 + 1$ . In addition, we choose

$$A^*(s) = (s+1) \left(s + \frac{2}{3}\right) \left(s + \frac{3}{4}\right) \left(s + \frac{4}{5}\right) \left(s + \frac{5}{6}\right). \quad (28)$$

**Specification of  $\theta_1^*$ ,  $\theta_2^*$  and  $\theta_3^*$ .** To obtain the parameters in the control law (8), we solve the Diophantine equation in (6). Specifically, let  $C(s) = s + c_0$ ,  $D(s) = d_3s^3 + d_2s^2 + d_1s + d_0$ . Then, substituting them into the following equation:

$$C(s)Q_m(s)P(s) + D(s)Z(s) = A^*(s), \quad (29)$$

we derive the unique solution as

$$\begin{aligned} C(s) &= s + \frac{67}{10}, \\ D(s) &= -\frac{73}{20}s^3 - \frac{293}{72}s^2 - \frac{541}{360}s + \frac{643}{120}. \end{aligned} \quad (30)$$

Choosing  $\Lambda_c(s) = (s+1)^3$ , from PPC law (8), we calculate the parameters  $\theta_1^*$  and  $\theta_2^*$  as

$$\begin{aligned} \theta_1^* &= \left(-\frac{57}{10}, 2, -\frac{37}{10}\right), \quad \theta_3^* = -\frac{73}{20}, \\ \theta_2^* &= \left(-\frac{41}{24}, \frac{3401}{360}, -\frac{2477}{360}\right). \end{aligned}$$

**Specification of  $M$ ,  $\Omega$  and  $\tau$ .** From (25) and (26), one can see that the choice of  $M$  needs to be balanced between the scaling factor  $\Omega$  and the switching time interval  $\tau$ . In this paper, we select  $\delta = 0.001$ . Then,  $M$  is chosen as 2367. Finally,  $\tau$  and  $\Omega$  are calculated from (25) and (26) as 24.8288 and 0.2356, respectively.

**Finite-and-quantized feedback PPC law.** In particular, with  $\Lambda_c(s) = (s+1)^3$ , the auxiliary systems that can generate the signals  $\omega_i(t)$ ,  $i = 1, 2, 3$  are

$$\begin{aligned} \dot{\omega}_1(t) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \omega_1(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t), \\ \dot{\omega}_2(t) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \omega_2(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega_3(t), \\ \dot{\omega}_3(t) &= Q(y(t), \Delta(t), y^*(t)) \Delta(t). \end{aligned}$$

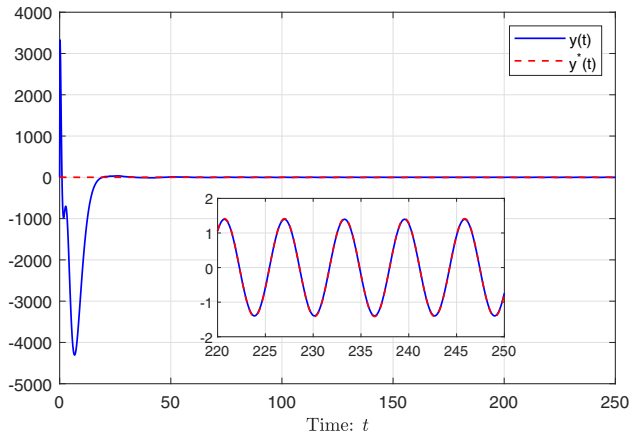


Fig. 2. Trajectories of output  $y(t)$  and reference output  $y^*(t)$  ( $y(0) = 2850$ ).

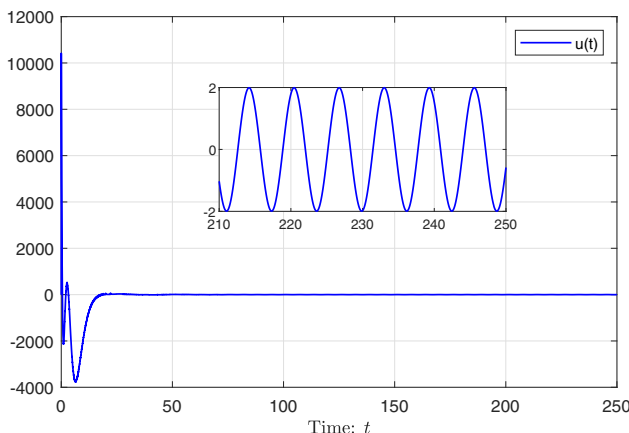


Fig. 3. Trajectory of the control law (8) ( $y(0) = 2850$ ).

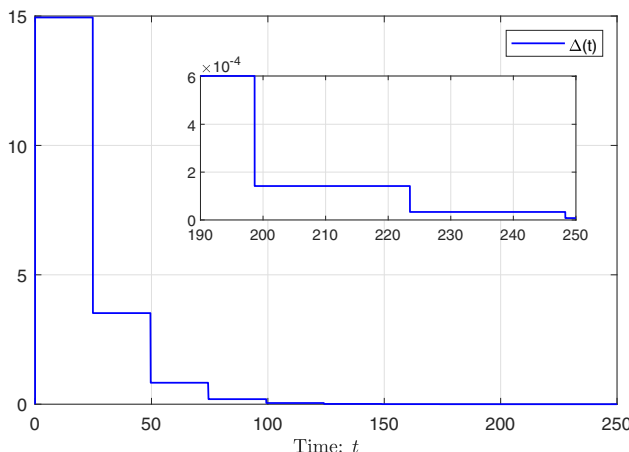


Fig. 4. Trajectory of  $\Delta(t)$  ( $y(0) = 2850$ ).

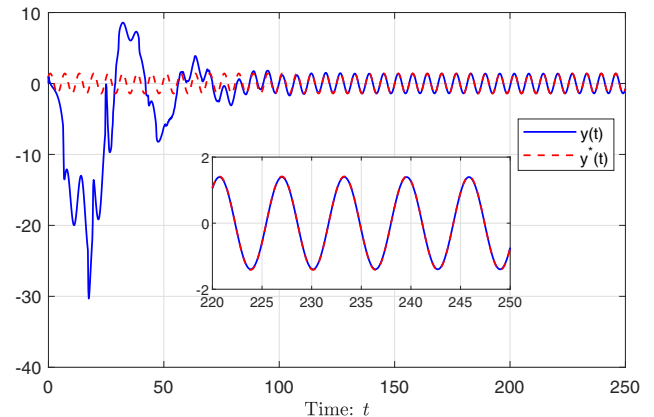


Fig. 5. Response of output  $y(t)$  vs. reference output  $y^*(t)$  ( $y(0) = 1$ ).

**Simulation figures.** We present two cases to validate the effectiveness of the proposed control method.

**Case 1.** Consider the case when  $y(0) = 2850$ . In this case, the initial value  $y(0)$  is larger than the saturation value  $M$ . The response of the system output  $y(t)$  to the reference output  $y^*(t)$  is illustrated in Fig. 2, indicating satisfactory tracking performance for  $t$  greater than approximately 50. The response of the finite-and-quantized output feedback PPC law is shown in Fig. 3. The temporal evolution of the sensitivity  $\Delta(t)$  is depicted in Fig. 4.

**Case 2.** Consider the case when  $y(0) = 1$ . The simulation results for this case are presented as follows: Fig. 5 depicts the response of the system output  $y(t)$  to the reference output  $y^*(t)$ . It can be observed from Fig. 5 that satisfactory tracking performance is achieved when  $t$  exceeds approximately 70. Additionally, Fig. 6 illustrates the response of the quantized output feedback PPC law, while Fig. 7 displays a graph of sensitivity  $\Delta(t)$  over time.

From Figs. 2–4 and Figs. 5–7, it can be seen that  $\Delta(t)$  scales down gradually at an exponential rate according to

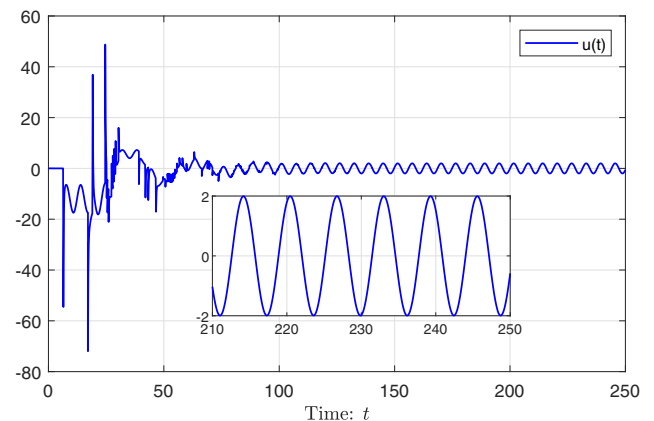


Fig. 6. Response of the control law (8) ( $y(0) = 1$ ).



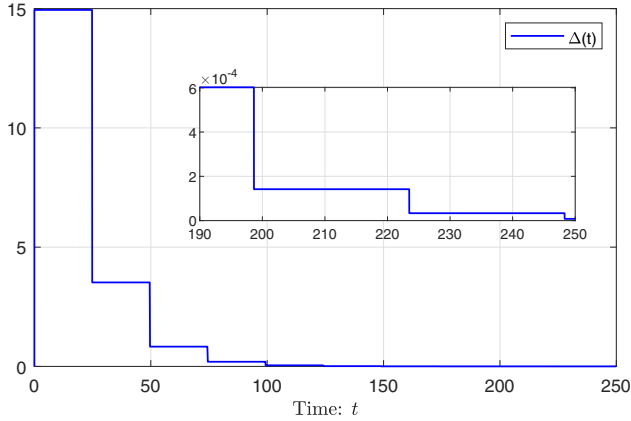


Fig. 7. Response of  $\Delta(t)$  ( $y(0) = 1$ ).

fixed time intervals, while the system output still manages to stay within the quantizer saturation range, so that asymptotic tracking is achieved. Figures 3 and 6 show that the control law is also stable. The simulation results are consistent with the theoretical findings. In conclusion, the simulation results not only validate the efficacy of the proposed method but also demonstrate its applicability to scenarios with different initial values. Moreover, taking into account the instability of  $P(s)$  and  $Z(s)$ , the simulation results further validate the efficacy of the proposed approach for nonminimum phase systems.

## 5. Conclusions

In this paper, we have developed a PPC-based solution to the asymptotic output tracking control problem for a class of continuous-time LTI systems by using finite-and-quantized output feedback. The controlled system allows for the presence of unstable poles and zeros, while still achieving closed-loop stability and asymptotic output tracking under the same conditions as the classical PPC method. Particularly, the developed solution provides a new methodology to adjust the sensitivity and thresholds for a particular class of dynamic quantizers, which is essential for asymptotic convergence of the output tracking error.

The following questions are of interest for further exploration: (i) How can adaptive control be implemented based on the proposed control method when the coefficients of  $P(s)$  and  $Z(s)$  are unknown? (ii) Is it possible to extend the proposed control method to nonlinear systems?

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## Appendix A

**Proof of Lemma 3.2** First, under the condition that no quantization error exists, i.e.

$$Q(y(t), \Delta(t), y^*(t)) \Delta(t) = y(t) - y^*(t),$$

the feedback control law (7) ensures that the tracking error  $e(t)$  converges to zero exponentially as  $t$  goes to infinity. The proof is similar to the proof of Lemma 1 in [44], so it is omitted here.

As for the quantized version, noting the formula (19), the finite-and-quantized output feedback PPC law structure (7) can be written as

$$\begin{aligned} u(t) &= (\Lambda_c(s) - C(s)Q_m(s)) \frac{1}{\Lambda_c(s)} [u](t) \\ &\quad - \left( \frac{D(s)}{\Lambda_c(s)} [y^* - y](t) + \frac{D(s)}{\Lambda_c(s)} [Q(y, \Delta, y^*) \Delta](t) \right) \\ &\quad + \frac{D(s)}{\Lambda_c(s)} [y^* - y](t) \\ &= (\Lambda_c(s) - C(s)Q_m(s)) \frac{1}{\Lambda_c(s)} [u](t) \\ &\quad + \frac{D(s)}{\Lambda_c(s)} [e_Q(y, \Delta, y^*)](t) + \frac{D(s)}{\Lambda_c(s)} [y^* - y](t), \end{aligned} \quad (\text{A.1})$$

which is equivalent to

$$\begin{aligned} C(s)Q_m(s)[u](t) - D(s)[y^* - y](t) \\ = D(s)[e_Q(y, \Delta, y^*)](t). \end{aligned} \quad (\text{A.2})$$

Multiplying both sides of this equation by  $Z(s)$ , we have

$$\begin{aligned} Z(s)C(s)Q_m(s)[u](t) - Z(s)D(s)[y^* - y](t) \\ = Z(s)D(s)[e_Q(y, \Delta, y^*)](t). \end{aligned}$$

Recalling that  $Q_m(s)[y^*](t) = 0$  in (5) and the system (2)  $P(s)y(t) = Z(s)[u](t)$ , we add to both sides of this equation by  $C(s)P(s)Q_m(s)[y^*](t)$ , and hence we can obtain

$$\begin{aligned} (C(s)Q_m(s)P(s) + D(s)Z(s))[e](t) \\ = Z(s)D(s)[e_Q(y, \Delta, y^*)](t), \end{aligned} \quad (\text{A.3})$$

that is

$$A^*(s)[e](t) = Z(s)D(s)[e_Q(y, \Delta, y^*)](t).$$

Since  $A^*(s)$  is stable, it follows that there exists some exponentially decaying  $\epsilon(t)$  such that (21) holds. Thus, the proof is completed.  $\square$

## Appendix B

**Proof of Theorem 3.6** From (20), consider the auxiliary system (22). The closed-loop system of (22) can be written as

$$\dot{x}(t) = Ax(t) + bQ_s(cx(t)), \quad (\text{B.1})$$

where  $Q_s(cx(t)) = cx(t) - \Delta(t)Q(y(t))$ . Noting the assumption that  $\|x(t_0)\| < U$  and that the sensitivity  $\Delta(t_0)$  is defined by

$$\Delta(t_0) \geq \|c\|U \left(M - \frac{1}{2}\right)^{-1} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}},$$

it follows that

$$\begin{aligned} x(t_0)^T Px(t_0) &\leq \lambda_{\max}(P)\|x(t_0)\|^2 \\ &\leq \lambda_{\max}(P)U^2 \\ &\leq \frac{1}{\|c\|^2} (\Delta(t_0))^2 \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P). \end{aligned} \quad (\text{B.2})$$

First we show that the quantizer defined by (3) does not saturate at time  $t_0$  if (B.2) holds. By the design of the quantizer (3), it is sufficient to prove

$$|y(t_0) - y^*(t_0)| \leq \left(M - \frac{1}{2}\right) \Delta(t_0). \quad (\text{B.3})$$

In fact, by  $x(t_0)^T Px(t_0) \geq \lambda_{\min}(P)\|x(t_0)\|^2$  and (B.2), it follows that

$$\|x(t_0)\| \leq \frac{1}{\|c\|} \left(M - \frac{1}{2}\right) \Delta(t_0). \quad (\text{B.4})$$

By (B.4) and  $y(t) - y^*(t) = cx(t)$ , we have

$$|y(t) - y^*(t)| \leq \|c\|\|x(t_0)\| \leq \left(M - \frac{1}{2}\right) \Delta(t_0).$$

This concludes that the quantizer does not saturate at  $t_0$ .

Second we investigate the properties of the Lyapunov function  $x(t)^T Px(t)$  when the quantizer is not saturated. In fact, if the quantizer is not saturated, we have

$$\|Q_s(cx(t))\| = \|Q_s(e(t))\| \leq \Delta(t)\sqrt{l}/2. \quad (\text{B.5})$$

Noting Eq. (23), if (B.3) holds, the derivative of  $x^T Px$  along the solutions of (B.1) is given by

$$\begin{aligned} \frac{d}{dt} x(t)^T Px(t) &= -x(t)^T x(t) + 2x(t)^T PbQ_s(cx(t)) \\ &\leq -\|x(t)\|^2 + 2\|x(t)\|\|Pb\|\Delta(t)\sqrt{l}/2 \\ &= -\|x(t)\|(\|x(t)\| - \|Pb\|\Delta(t)\sqrt{l}). \end{aligned} \quad (\text{B.6})$$

This means the derivative of  $x^T Px$  is negative outside the set  $\{x : \|x\| \leq \|Pb\|\Delta(t)\sqrt{l}\}$ .

For an arbitrary  $\delta > 0$ , we choose  $M$  satisfying

$$M > \|c\|(\|Pb\|\sqrt{l} + \delta) \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + 1/2.$$

Recall the definition of the scaling factor  $\Omega$  and the switching interval  $\tau$ :

$$\begin{aligned} \Omega &= \|c\|(\|Pb\|\sqrt{l} + \delta) \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(M - \frac{1}{2}\right)^{-1}, \\ \tau &= \frac{\frac{1}{\|c\|^2} (M - \frac{1}{2})^2 \lambda_{\min}(P) - (\|Pb\|\sqrt{l} + \delta)^2 \lambda_{\max}(P)}{(\|Pb\|\sqrt{l} + \delta)\delta}. \end{aligned}$$

For the ‘‘zooming in’’ stage, in the first  $\tau$  interval, the sensitivity  $\Delta(t)$  is equal to  $\Delta(t_0)$ . We then show that the quantizer still does not saturate when  $t_0 < t \leq t_0 + \tau$ . Define the following set:

$$R = \left\{ x : x^T Px \leq \frac{1}{\|c\|^2} (\Delta(t_0))^2 \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P) \right\}.$$

The sufficiency of proving that the quantizer still does not saturate lies in demonstrating that  $x(t)$  will not exceed the range  $R$ . Since  $\Omega < 1$  and  $\delta > 0$ , we have

$$\lambda_{\max}(P)\|Pb\|^2 l < \frac{1}{\|c\|^2} \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P). \quad (\text{B.7})$$

If  $\|x(t)\| > \|Pb\|(\Delta(t_0))\sqrt{l}$ , then by (B.6),  $x(t)^T Px(t)$  decreases and thus  $x(t)$  will not leave  $R$ . If  $\|x(t)\| \leq \|Pb\|(\Delta(t_0))\sqrt{l}$ , then by (B.7), we have

$$\begin{aligned} x(t)^T Px(t) &\leq \lambda_{\max}(P)\|x(t)\|^2 \\ &\leq \lambda_{\max}(P)\|Pb\|^2 (\Delta(t_0))^2 l \\ &\leq \frac{1}{\|c\|^2} \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P) (\Delta(t_0))^2, \end{aligned}$$

which implies  $x(t)$  does not leave  $R$ . Therefore, the quantizer still does not saturate when  $t_0 < t \leq t_0 + \tau$ .

Next, we prove that

$$\begin{aligned} x^T(t_0 + \tau)Px(t_0 + \tau) &\leq (\Delta(t_0))^2 (\|Pb\|\sqrt{l} + \delta)^2 \lambda_{\max}(P) \\ &= (\Omega \Delta(t_0))^2 \frac{1}{\|c\|^2} \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P). \end{aligned} \quad (\text{B.8})$$

Assume that (B.8) is false, i.e.

$$\begin{aligned} x^T(t_0 + \tau)Px(t_0 + \tau) &> (\Delta(t_0))^2 (\|Pb\|\sqrt{l} + \delta)^2 \lambda_{\max}(P). \end{aligned} \quad (\text{B.9})$$

By  $x^T(t_0 + \tau)Px(t_0 + \tau) \leq \lambda_{\max}(P)\|x(t)\|^2$  and (B.9), we have

$$\|x(t_0 + \tau)\| > \Delta(t_0)(\|Pb\|\sqrt{l} + \delta). \quad (\text{B.10})$$

Combing (23) and (B.10), it follows that

$$\|x(t)\| > \Delta(t_0)(\|Pb\|\sqrt{I} + \delta), \quad \forall t \in [t_0, t_0 + \tau]. \quad (\text{B.11})$$

Bringing (B.11) into (B.6), we have

$$\frac{d}{dt}x(t)^T Px(t) \leq -\Delta(t_0)^2(\|Pb\|\sqrt{I} + \delta)\delta. \quad (\text{B.12})$$

The comparison of the last inequality with (26), (B.2) and (B.9) inevitably leads to a contradiction which proves that (B.8) holds.

Noting Eq. (B.8), at the moment  $t_0 + \tau < t < t_0 + 2\tau$ , we make the sensitivity in the control law (8) shrink to  $\Delta(t) = \Omega\Delta(t_0)$ . Using the same analysis as before we obtain

$$\begin{aligned} & x^T(t_0 + \tau)Px(t_0 + \tau) \\ & \leq (\Omega^2\Delta(t_0))^2 \frac{1}{\|c\|^2} \left(M - \frac{1}{2}\right)^2 \lambda_{\min}(P). \end{aligned}$$

The same can be done at the moment  $t_0 + 2\tau < t < t_0 + 3\tau$ , such that the sensitivity in the the control law (8) reduces to  $\Delta(t) = \Omega^2\Delta(t_0)$ .

Since the previous operation does not cause the quantizer to saturate, the quantization error by equation (B.5) does not exceed  $\Delta(t)\sqrt{I}/2$  and the tracking error does not exceed  $(M - 1/2)\Delta(t)$ . Repeating the above steps and note that  $\Omega < 1$ , we conclude that the tracking error will gradually converge to zero. Also the boundedness of the output  $y(t)$  can be immediately obtained from the boundedness of the tracked signal  $y^*(t)$ . Finally we show the boundedness of the input  $u(t)$ . Operating both sides of (6) on  $u(t)$ , we have

$$C(s)Q_m(s)P(s)[u](t) + D(s)Z(s)[u](t) = A^*(s)[u](t). \quad (\text{B.13})$$


Then, substituting (2) and (A.2) into (B.13), it follows that


$$A^*(s)[u](t) = P(s)D(s)[y^*](t) + P(s)D(s)[e_Q(y, \Delta, y^*)](t). \quad (\text{B.14})$$

From the previous analysis we know that the steps of the method proposed in this paper enable the quantizer to remain unsaturated, i.e.  $e_Q(y(t), \Delta(t), y^*(t)) \leq \Delta(t)\sqrt{I}/2$ , and this result, combined with the boundedness of  $y^*(t)$ , equation (B.14), and the stability of  $A^*(s)$ , yields  $u(t)$  to be bounded. This completes the proof.  $\square$

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